

KEMER'S THEORY FOR H -MODULE ALGEBRAS WITH APPLICATION TO THE PI EXPONENT

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ABSTRACT. Let H be a semisimple finite dimensional Hopf algebra over a field F of zero characteristic. We prove three major theorems:. 1. The Representability theorem which states that every H -module (associative) F -algebra W satisfying an ordinary PI, has the same H -identities as the Grassmann envelope of an $H \otimes (F\mathbb{Z}/2\mathbb{Z})^*$ -module algebra which is finite dimensional over a field extension of F . 2. The Specht problem for H -module (ordinary) PI algebras. That is, every H - T -ideal Γ which contains an ordinary PI contains H -polynomials f_1, \dots, f_s which generates Γ as an H - T -ideal. 3. Amitsur's conjecture for H -module algebras, saying that the exponent of the H -codimension sequence of an ordinary PI H -module algebra is an integer.

1. INTRODUCTION

Two of the main problems in the theory of associative algebras satisfying a polynomial identity (PI in short) are the Specht problem (see [13]) and the Representability theorem ([11]). The classical Specht problem asks whether a T -ideal can be generated as a T -ideal by a finite number of polynomials. The Representability theorem states that every PI algebra has the same identities (PI equivalent) as the Grassmann envelope of a $\mathbb{Z}/2\mathbb{Z}$ -graded finite dimensional algebra. Moreover, if the given PI algebra is affine, then it is PI equivalent to a finite dimensional algebra. The two theorems seems unrelated, since there is no obvious reason for a T -ideal of identities of (even) a finite dimensional algebra to be finitely based. However, both of them were solved in the 80's by Kemer [11] using the same ideas. Thus intertwining the two problems.

In the recent decades different classes of algebras, such as non-associative algebras, group graded algebras, group acted algebras, algebras with involution, were studied in the context of PI theory. In all of these frameworks analogs of these problems exist and in some of them also solved: For finite group-graded algebras satisfying an ordinary PI see [3] (it is worth mentioning that in [15] the special case of abelian finite groups is treated). For algebras with involutions satisfying an ordinary PI see [16]. For affine algebras over fields of non-zero characteristic see [1]. The assumption that the algebra satisfies an ordinary PI (and not just a PI of the framework in consideration) is essential for the Representability theorem, since finite dimensional algebras and the Grassmann envelope of a finite dimensional algebras are satisfying an ordinary PI. However, it might be the case that the Specht problem remains true without this assumption.

In this paper we work in the framework of H -module algebras satisfying an ordinary PI, where H is a finite dimensional and semisimple Hopf F -algebra (F is a characteristic

Key words and phrases. graded algebra, polynomial identity, Hopf algebra, exponent.

zero field). Two important examples of families of algebras which this framework generalizes are the (finite) group graded algebras and the group acted algebras: Suppose G is any finite group. By considering H to be the dual of the group algebra FG we obtain the family of G -graded algebras; whereas by considering $H = FG$ we obtain the family of algebras with a G action (by F -algebra automorphisms). So far the Specht and Representability problems were open for the latter family in the case where G is non-abelian. If G is abelian, then these problems are equivalent to the corresponding problems in the G -graded case (same G).

Let us introduce the notation to discuss these problems. Suppose H is an m -dimensional Hopf algebra over a field F of characteristic zero and let W be an H -module algebra over F . Suppose $X = \{x_1, \dots, x_n, \dots\}$ is a set of non-commutative variables and consider the vector space $V = FX \otimes_F H$. An H -polynomial is an element in the tensor algebra (without 1) over V , which we denote by $F^H \langle X \rangle$. One might prefer instead a coordinate oriented definition of $F^H \langle X \rangle$: Choose a basis $\{b_1, \dots, b_m\}$ for the F -algebra H . Then $F^H \langle X \rangle$ is understood as the F -algebra generated by the formal (non-commutative) variables x^{b_i} , where $i \in \{1, \dots, m\}$ and $x \in X$. Notice that $F^H \langle X \rangle$ is an H -module algebra, where

$$h \cdot ((x_{i_1} \otimes h_1) \otimes \dots \otimes (x_{i_k} \otimes h_k)) = (x_{i_1} \otimes h_{(1)}h_1) \otimes \dots \otimes (x_{i_k} \otimes h_{(k)}h_k)$$

or

$$h \cdot x_{i_1}^{h_1} \dots x_{i_k}^{h_k} = x_{i_1}^{h_{(1)}h_1} \dots x_{i_k}^{h_{(k)}h_k},$$

where $h_1, \dots, h_k \in H$ (we use the Swidler notation: $\Delta(h) = h_{(1)} \otimes h_{(2)}$).

We say that $f \in F^H \langle X \rangle$ is an identity of W if for every H -homomorphism $\phi : F^H \langle X \rangle \rightarrow W$ the polynomial f is in the kernel of ϕ . Put differently, f is an identity of W if f vanishes for every substitution of the variables from X by elements of W . The set of all identities, denoted by $id^H(W)$, is an ideal of $F^H \langle X \rangle$ which is also stable under H -endomorphisms. Such an ideal is called H - T -ideal.

Finally, suppose W_1 and W_2 are two H -module F -algebras. We say that $W_1 \sim_{H-PI} W_2$ (H -PI equivalent) if $id^H(W_1) = id^H(W_2)$. It is crucial to notice that $W \sim_{H-PI} \mathcal{W}$, where \mathcal{W} (always) denotes the relatively free H -module algebra $F^H \langle X \rangle / id^H(W)$.

The main part of this paper is dedicated to proving the following theorem:

Theorem 1.1 (Affine H -Representability). *Let W be an affine H -module algebra over a field F of characteristic zero satisfying an ordinary polynomial identity, where H is a finite dimensional semisimple Hopf F -algebra. Then there exists a field extension L of F and a finite dimensional H -module algebra A over L which is H -PI equivalent to W .*

To state the general H -representability theorem we need more notations. Denote by $E = E_0 \oplus E_1$ the Grassmann superalgebra over F . Suppose W is an $H_2 = H \otimes_F (F\mathbb{Z}/2\mathbb{Z})^*$ -module algebra. In other words, $W = W_0 \oplus W_1$ is a superalgebra endowed with H -module algebras structure such that W_0 and W_1 are stable under the action of H . The Grassmann envelope of W is the H_2 -module F -algebra $E(W) = (W_0 \otimes E_0) \oplus (W_1 \otimes E_1)$. The H -representability theorem states:

Theorem 1.2 (H-Representability). *Let W be an H -module algebra over a field F of characteristic zero satisfying an ordinary polynomial identity, where H is a finite dimensional semisimple Hopf F -algebra. Then there exists a field extension L of F and a finite dimensional H_2 -module algebra A over L such that $W \sim_{H-PI} E(A)$.*

In the final section of this paper we obtain:

Theorem 1.3 (Specht). *Suppose Γ is an H - T -ideal containing an ordinary identity, then there are $f_1, \dots, f_s \in \Gamma$ which H - T -generate Γ . Equivalently, if $\Gamma_1 \subseteq \Gamma_2 \subseteq \dots$ is an ascending chain of H - T -ideals containing an ordinary PI, then the chain stabilizes.*

The first and main part of this article is the proof of theorem 1.1. For this we follow, for the most part, the exposition of Kemer's proof given in [4]. However, there are two major differences. The first is the proof of "Kemer Lemma 1" and the second is the construction of "representable spaces" for the Kemer polynomials (see section §3 for details). The conclusion of theorem 1.2 and theorem 1.3 is completely standard and we use the same argument as in [3, 10].

Let us recall the definition of the H -codimension sequence of an H -module algebra:

Definition 1.4. Let W be an H -module F -algebra. The H -codimension of W is

$$c_n^H(W) = \dim_F P_n^H / P_n^H \cap id^H(W),$$

where P_n^H is the F -space spanned by $x_{\sigma(1)}^{h_1} \cdots x_{\sigma(n)}^{h_n}$, where $\sigma \in S_n$ and $h_1, \dots, h_n \in H$.

A consequence of theorem 1.2 is the affirmative solution of Amitsur's conjecture on the exponent in the case of general H -module F -algebras.

Theorem 1.5 (H -Amitsur's Conjecture). *Suppose W is any H -module F -algebra which satisfies an ordinary PI, then the H -exponent of W defined by*

$$\exp^H(W) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^H(W)}$$

exists and is an integer.

Using the ideas of Gordianko and Zaicev in [8] and Gordienko in [9] we will obtain this theorem in the final section.

2. PRELIMINARIES

There are two families of H -polynomials which play a leading role in PI theory: multilinear and alternating polynomials:

Definition 2.1. $f(x_1, \dots, x_n) \in F^H\{X\}$ is *multilinear* if

$$f(x_1, \dots, x_{i-1}, \alpha x_i + y, x_{i+1}, \dots, x_n) = \alpha f(x_1, \dots, x_n) + f(x_1, \dots, x_{i-1}, y, x_i, \dots, x_n)$$

for every i between 1 to n and $\alpha \in F$.

Remark 2.2. In the case where H is the dual of the group algebra FG (here G is a finite group) the polynomial:

$$f(x, y) = x_g y + y_g x_e$$

is multilinear by our definition.

Definition 2.3. Let $f = f(x_1, \dots, x_n) \in F^H \{X\}$. For $1 \leq i, j \leq n$ we denote by $f|_{x_i \rightarrow x_j} = f|_{x_i=x_j}$ the polynomial obtained from f by substituting x_i inside x_j . Moreover, $f|_{x_i \leftrightarrow x_j}$ denotes the polynomial obtained from f by replacing x_i by x_j and vice versa.

Definition 2.4. Suppose $f = f(x_1, \dots, x_n, Y) \in F^H \{X\}$, where Y is a set of variables disjoint from x_1, \dots, x_n . We say that f is alternating on x_1, \dots, x_n if

$$f|_{x_i \leftrightarrow x_j} = -f$$

for every i and j between 1 to n . Since the characteristic of F is not 2 this is equivalent to

$$f|_{x_i=x_j} = 0.$$

If $f = f(X, Y) \in F^H \{X\}$ is any polynomial we define

$$\text{Alt}_X(f) = \sum_{\sigma \in S_X} (-1)^\sigma f|_{x \in X \leftarrow \sigma(x)}.$$

Therefore, $\text{Alt}_X(f)$ is alternating on X . If f was alternating on X to begin with, then $\text{Alt}_X(f) = |X|! \cdot f$.

Remark 2.5. As in classical PI theory any H - T -ideal Γ is T -generated by the multilinear polynomials inside Γ .

Finally, suppose W_1 and W_2 are two H -module F -algebras. We say that $W_1 \sim_{H-PI} W_2$ (H -PI equivalent) if $\text{id}^H(W_1) = \text{id}^H(W_2)$. It is crucial to notice that $W \sim_{H-PI} \mathcal{W}$, where \mathcal{W} (always) denotes the relatively free H -module algebra $F^H \{X\} / \text{id}^H(W)$.

3. SKETCH OF THE PROOF OF THEOREM 1.1

In this short section we outline the main steps of the proof of theorem 1.1.

- (1) Every affine (ordinary) PI H -module F -algebra W has a finite dimensional H -module F -algebra A such that $\text{id}^H(A) \subseteq \text{id}^H(W)$.
- (2) Definition of the H -Kemer index $\text{Ind}(\Gamma) = (\alpha, r) \in \Omega = \mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$ and H -Kemer polynomials for H T -ideals of H -module algebras satisfying some Capelli identity. Since by the previous step any affine PI H -module algebra satisfies a Capelli identity, the index is defined for all the algebras under consideration.

Considering the lexicographic ordering (\leq) on Ω it will be easy to conclude that if $\Gamma_1 \subseteq \Gamma_2$ then $\text{Ind}(\Gamma_1) \leq \text{Ind}(\Gamma_2)$ (reverse ordering).

- (3) Construction of H -basic algebras. Every H -basic H -module algebra A is finite dimensional and has the property $\text{Ind}(A) = \text{Par}(A) = (d, s-1)$, where d is the dimension of the semisimple part of A and s is the nilpotency of $J(A)$, the radical of A . We show that every finite dimensional H -module algebra is H -PI equivalent to a finite direct product of H -basic algebras. As far as the author knows, this step in all other frameworks (e.g. group graded algebras, algebras with involutions) relies heavily on precise knowledge of all the simple, finite dimensional objects of the category in question (see [4, 3]). However, in such general framework as H -module algebras it seems that one must consider more

“subtle” approach. Luckily, such approach was already introduced for different purpose by Gordienko in [9].

- (4) There is a finite dimensional H -module algebra B having the same H -Kemer index and H -Kemer polynomials as W .
- (5) Using steps 3 and 4 the Phoenix property for H - T -ideals will follow. This property states that if $f \notin \Gamma$ is a consequence of an H -Kemer polynomial of Γ , then although f might fail being an H -Kemer polynomial, yet it has a consequence f' which is an H -Kemer polynomial of Γ .
- (6) Construction of a **representable** H -module algebra B_Γ satisfying the properties:
 - $id^H(B_\Gamma) \supseteq \Gamma$.
 - All H -Kemer polynomials of Γ are non-identities of B_Γ .
- (7) We finalize the proof. consider $\Gamma' = \Gamma + S$, where S is the H - T -ideal *generated* by all H -Kemer polynomials of Γ . This will imply that $\text{Ind}(\Gamma') < \text{Ind}(\Gamma)$ and hence by induction on the H -Kemer index there exists a finite dimensional H -module algebra A' with $\Gamma' = id^H(A')$. We show that *all* polynomials of S (which are not in Γ) are nonidentities of B_Γ (that is, not just elements in S which are H -Kemer polynomials). This is achieved by the Phoenix property of Kemer polynomials. Since any nonidentity f' of Γ which is in S , produces (by the T -operation) a Kemer polynomial which by Step 5 is not in $id^H(B_\Gamma)$ we have also that $f' \notin id^H(B_\Gamma)$. From that one concludes that $\Gamma = id^H(A' + B_\Gamma)$.

4. GETTING STARTED

Theorem 4.1. *Suppose W is an affine H -module algebra which satisfies an ordinary PI, then there is a finite dimensional H -module F -algebra A such that $id^H(A) \subseteq id^H(W)$.*

Proof. By the classical PI theory (see Corollary 4.9 in [10]) there is an F -algebra A_0 with the property $id(A_0) \subseteq id(W)$. Consider the H -module algebra $A = A_0 \otimes H^*$, where the H -action is given by

$$h(a \otimes \phi) = a \otimes \phi_h, \quad \phi_h(g) = \phi(gh) \quad g, h \in H, \phi \in H^*.$$

Since

$$\begin{aligned} (h(\phi \cdot \psi))(g) &= (\phi \cdot \psi)(gh) = \phi(g_{(1)}h_{(1)})\psi(g_{(2)}h_{(2)}) = \phi_{h_{(1)}}(g_{(1)})\phi_{h_{(2)}}(g_{(2)}) \\ &= (\phi_{h_{(1)}} \cdot \phi_{h_{(2)}})(g) = (h_{(1)}\phi \cdot h_{(2)}\psi)(g) \end{aligned}$$

and

$$(h \cdot \epsilon)(g) = \epsilon(gh) = \epsilon(h)\epsilon(g),$$

we indeed defined an H -action.

Suppose $f = f(x_1, \dots, x_n) \in id^H(A)$. We need to show that $f \in id^H(W)$. Let $\phi \in H^*$ be defined by $\phi(h_i) = \delta_{1,i} \in F$, where $i = 1, \dots, m = \dim_F H$. The important property of ϕ is that $h_1 \cdot \phi, \dots, h_m \cdot \phi$ are linearly independent over F . Consider the substitution $\bar{x}_1 = a_1 \otimes \phi, \bar{x}_2 = a_2 \otimes 1, \dots, \bar{x}_n = a_n \otimes 1$ (here 1 is the functional of H which equals to

1 at every point). We obtain

$$f(\bar{x}_1, \dots, \bar{x}_n) = \sum_{i=1}^m g_i(a_1, \dots, a_n) \otimes h_i \cdot \phi,$$

where $g_i \in F^H\{X\}$ is multilinear polynomial all of whose monomials contain the variable $x_1^{h_i}$. Therefore, $g_i \in id^H(A)$. We may replace $x_1^{h_i}$ by x_1 and obtain $g_i^{(1)} \in id^H(A)$. Notice that it suffices to show that $g_1^{(1)}, \dots, g_m^{(1)} \in id^H(W)$.

Repeat the argument for each one of the polynomials $g_1^{(1)}, \dots, g_m^{(1)}$, by considering the substitution $\bar{x}_1 = a_1 \otimes 1, \bar{x}_2 = a_2 \otimes \phi, \bar{x}_3 = a_3 \otimes 1, \dots, \bar{x}_n = a_n \otimes 1$. This will result in multilinear polynomials $g_1^{(2)}, \dots, g_m^{(2)} \in id^H(A)$ having the properties:

- All the monomials of each $g_i^{(2)}$ contain x_1 and x_2 .
- If $g_1^{(2)}, \dots, g_m^{(2)} \in id^H(A)$, then $f \in id^H(A)$.

Repeating this argument eventually results in the conclusion that $f \in id^H(W)$ if and only if some (ordinary!) polynomials $g_1^{(n)}, \dots, g_m^{(n)} \in id(A)$ are in $id(W)$. However, this indeed holds due to the assumption on A . □

Definition 4.2. Let W be an H -module F -algebra. We say that W satisfies a Capelli identity m if every H -polynomial $f(x_1, \dots, x_m, Y)$ which is alternating in x_1, \dots, x_m is in $id^H(W)$.

The following definition of H -Kemer index and H -Kemer polynomials makes sense only for H -module algebras satisfying a Capelli identity. As we saw previously, this includes the affine H -module algebras which satisfy an ordinary PI.

Definition 4.3. Suppose Γ satisfies some Capelli identity. Define $\alpha(\Gamma)$ to be the maximal integer such that for every μ there is a multilinear polynomial $f = f(X_1, \dots, X_\mu, Y) \notin \Gamma$ which is alternating with respect to the sets X_1, \dots, X_μ which are all of cardinality $\alpha(\Gamma)$.

$s(\Gamma)$ is defined as the maximal integer such that for every ν there is a multilinear $g = g(X_1, \dots, X_\mu, X'_1, \dots, X'_{s(\Gamma)}, Y) \notin \Gamma$ which is alternating with respect to $X_1, \dots, X_\mu, X'_1, \dots, X'_{s(\Gamma)}$, where $|X_1| = \dots = |X_\mu| = \alpha(\Gamma)$ and $|X'_1| = \dots = |X'_{s(\Gamma)}| = \alpha(\Gamma) + 1$.

We call the pair $(\alpha(\Gamma), s(\Gamma))$ the H -Kemer index of Γ and denote it by $\text{Ind}(\Gamma)$. Any such g is called H -Kemer polynomial of Γ of rank μ . We refer to X_1, \dots, X_μ as *small sets* and to $X'_1, \dots, X'_{s(\Gamma)}$ as *big sets*.

Remark 4.4. If $\Gamma_1 \subseteq \Gamma_2$ then $\text{Ind}(\Gamma_1) \geq \text{Ind}(\Gamma_2)$ i.e. the order is reversed.

Remark 4.5. In what follows we will always assume that $\mu \geq \mu_\Gamma$ where μ_Γ is the minimal integer for which any multilinear $f = f(X_1, \dots, X_{\mu_\Gamma}, X'_1, \dots, X'_{s(\Gamma)+1}, Y) \in F^H\{X\}$, which alternates on $X_1, \dots, X_{\mu_\Gamma}, X'_1, \dots, X'_{s(\Gamma)}$, and $|X_1| = \dots = |X_{\mu_\Gamma}| = \alpha(\Gamma)$, $|X'_1| = \dots = |X'_{s(\Gamma)+1}| = \alpha(\Gamma) + 1$, is an identity of Γ .

5. THE INDEX OF FINITE DIMENSIONAL ALGEBRAS

We start this section with the definition of the Phoenix property.

Definition 5.1. (The Phoenix property) Let Γ be an H - T -ideal as above. Let P be any property which may be satisfied by polynomials (e.g. being H -Kemer). We say that P is “ Γ -Phoenix” (or in short “Phoenix”) if given a multilinear polynomial f having P which is not in Γ and *any* f' in $\langle f \rangle_H$ (the H - T -ideal generated by f) which is not in Γ as well, there exists a multilinear polynomial f'' in $\langle f' \rangle_H$ which is not in Γ and satisfies P . We say that P is “strictly Γ -Phoenix” if any multilinear polynomial $f' \in \langle f \rangle_H$ which is not in Γ , satisfies P .

Remark 5.2. Given a polynomial g , there exists a multilinear polynomial f' such that $\langle f' \rangle_H = \langle g \rangle_H$. It follows that in order to verify the Phoenix property it is sufficient to consider multilinear polynomials f' in $\langle f \rangle_H$.

Let us pause for a moment and summarize what we have at this point. We are given an H - T -ideal Γ (the T -ideal of identities of an affine H -module algebra W). We assume that W is ordinary PI and hence as shown in section §4 there exists a finite dimensional H -module algebra A with $\Gamma \supseteq id^H(A)$. To the H - T -ideal Γ we attach the corresponding H -Kemer index in $\mathbb{Z}^{\geq 0} \times \mathbb{Z}^{\geq 0}$. Similarly, we may consider the Kemer index of $id^H(A)$ which by abuse of notation we denote it by $\text{Ind}(A)$. Clearly, we have $\text{Ind}(\Gamma) \leq \text{Ind}(A)$.

One of our main goals (in the first part of the proof) is to replace the H -module algebra A by an H -module algebra A' with a larger T -ideal such that

- (1) $\Gamma \supseteq id^H(A')$
- (2) Γ and $id^H(A')$ have the same H -Kemer index.
- (3) Γ and $id^H(A')$ have the “same” H -Kemer polynomials.

Remark 5.3. The terminology “the same H -Kemer polynomials” needs a clarification. If $\Gamma_1 \supseteq \Gamma_2$ are $H - T$ ideals with $\text{Ind}(\Gamma_1) = \text{Ind}(\Gamma_2)$. We say that Γ_1 and Γ_2 have the same H -Kemer polynomials if there exists an integer μ such that all Kemer polynomials of Γ_2 with at least μ alternating small sets are not in Γ_1 . Write $\mu_{\Gamma, \Gamma'}$ for the maximum between the above μ , μ_Γ and $\mu_{\Gamma'}$.

Remark 5.4. Statements (1) – (3) above will establish the important connection between the combinatorics of the H -Kemer polynomials of Γ and the structure of finite dimensional H -module algebras. The “Phoenix” property for the H -Kemer polynomials of Γ will follow from that connection.

Let A be a finite dimensional H -module algebra over F and let $J(A)$ be its Jacobson radical. We know ([12]) that $J(A)$ is H -invariant, thus $\overline{A} = A/J(A)$ is a semisimple H -module algebra. Moreover by the H -invariant Wedderburn-Malcev Principal Theorem (see [14]) there exists a semisimple H -module subalgebra \overline{A} of A such that $A = \overline{A} \oplus J(A)$ as vector spaces. In addition, the subalgebra \overline{A} may be decomposed as an algebra into the direct product of H -simple algebras $\overline{A} \cong A_1 \times A_2 \times \cdots \times A_q$ (see [[8], Lemma 3]).

Remark 5.5. This decomposition enables us to consider “semisimple” and “radical” substitutions. More precisely, since in order to check whether a given multilinear H -polynomial is an identity of A it is sufficient to evaluate the variables on any (given)

spanning set, we may take a basis consisting of elements of $\overline{A} \cup J(A)$. We refer to such evaluations as semisimple or radical evaluations respectively. Moreover, the semisimple substitutions may be taken from the simple components.

In what follows, whenever we evaluate a polynomial on a finite dimensional H -module algebra, we consider only evaluations of that kind.

For any finite dimensional H -module algebra A over F we let $d(A)$ be the dimension of the semisimple subalgebra and n_A the nilpotency index of $J(A)$. We denote by $\text{Par}(A) = (d(A), n_A - 1)$ the *parameter* of the H -module algebra A .

Proposition 5.6. *Let (α, s) be the index of A . Then $(\alpha, s) \leq (d(A), n_A - 1)$.*

Proof. By the definition of the parameter α , there exist nonidentity polynomials with arbitrary large number of alternating sets of cardinality α . Now, if $\alpha > d(A)$ any such alternating set must have at least one radical evaluation and hence the polynomial cannot have more than $(n_A - 1)$ alternating sets of cardinality α . Contradiction. This shows $\alpha \leq d(A)$. In order to complete the proof of the proposition we need to see that if $\alpha = d(A)$ then $s < n_A$. To this end, recall that s is the maximal number of alternating sets of cardinality $\alpha + 1$ in nonidentities (in addition to arbitrary many alternating sets of cardinality α). But if $\alpha = d(A)$, then alternating sets of cardinality $\alpha + 1$ must contain at least one radical evaluation on any nonzero evaluation of its variables and hence, as above, the polynomial cannot contain more than $(n_A - 1)$ alternating sets of cardinality $\alpha + 1$. This proves the proposition. \square

In order to establish a precise relation between the index of a finite dimensional H -module algebra A and its structure we need to find appropriate finite dimensional H -module algebras which will serve as a **minimal model** for a given H -Kemer index. Here is the precise definition.

Definition 5.7. A finite dimensional H -module algebra A is said to be *H -PI-basic* (or just *H -basic*) if there are no finite dimensional H -module algebras B_1, \dots, B_s such that $\text{Par}(B_i) < \text{Par}(A)$ and A is H -PI equivalent to $B_1 \times \dots \times B_s$.

Remark 5.8. By induction on $\text{Par}(A)$ it is easy to see that every finite dimensional H -module algebra is H -PI equivalent to a finite product of H -basic algebras.

We need to understand what “PI properties” does H -basic algebras possess.

Definition 5.9. We say that a finite dimensional H -module algebra A is *full* with respect to a multilinear H -polynomial f , if exist a nonvanishing evaluation of f on A such that every H -simple component is represented (among the semisimple substitutions). A finite dimensional H -module algebra A is said to be full if it is full with respect to some multilinear H -polynomial f .

Lemma 5.10. *Let A be a finite dimensional H -module algebra which is not full. Then A is not H -basic.*

Proof. Since any H -module algebra with one H -simple component is full we may assume that $q > 1$. Consider the decompositions mentioned above $A \cong \bar{A} \oplus J$ and $\bar{A} \cong A_1 \times A_2 \times \cdots \times A_q$ (A_i are H -simple algebras). Construct the H -module subalgebras $B_i = (A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_q) \oplus J = \pi^{-1}(A_1 \times \cdots \times A_{i-1} \times A_{i+1} \times \cdots \times A_q)$, where $\pi : A \rightarrow \bar{A}$ is the natural projection.

We claim that the algebras A and $\tilde{A} = B_1 \times \cdots \times B_q$ are H -PI-equivalent: Of course $id^H(A) \subseteq id^H(\tilde{A})$, so it suffices to prove that any H -nonidentity f of A is also a nonidentity of \tilde{A} . Clearly, we may assume that f is multilinear (say of degree n). Consider a non zero evaluation $\bar{x}_1, \dots, \bar{x}_n$ of f on A . By assumption, there is some i such that $\bar{x}_1, \dots, \bar{x}_n \notin A_i$ so $\bar{x}_1, \dots, \bar{x}_n \in B_i$. Hence f is non zero on \tilde{A} . Since for every i $\text{Par}(B_i) < \text{Par}(A)$ we are done. \square

Proposition 5.11. *Let A be a finite dimensional H -module algebra which is full. Let $\text{Ind}(A) = (\alpha, s)$ and $\text{Par}(A) = (d(A), n_A - 1)$. Then $\alpha = d(A)$.*

For the proof we need to show that for an arbitrary large integer μ there exists a multilinear H -nonidentity f that contains μ folds of alternating sets of cardinality $\dim_F(\bar{A})$.

Lemma 5.12 (Kemer's Lemma 1). *Notation as above. Let A be a finite H -module dimensional algebra which is full. Then for any integer μ there exists a polynomial f in the T -ideal with the following properties:*

- (1) $f \notin id^H(A)$
- (2) f has μ -folds of alternating sets of cardinality $\dim_F(\bar{A})$.

Proof. See Lemma 10 in [9]. \square

6. KEMER'S LEMMA 2

In this section we prove Kemer's Lemma 2. Before stating the precise statement we need to extract an additional "PI property" from H -basic algebras. This time we need a property which controls the nilpotency index.

Let f be a multilinear H -polynomial which is not in $id^H(A)$. Clearly, any nonzero evaluation cannot have more than $n_A - 1$ radical evaluations.

Lemma 6.1. *Let A be a finite dimensional H -module algebra. Let $\text{Ind}(A) = (\alpha, s)$ be its Kemer index. Then $s \leq n_A - 1$.*

Proof. A is H -PI equivalent to the direct product of H -module algebras $B_1 \times \cdots \times B_q$, where B_i is full for $i = 1 \dots q$. For each B_i we consider the dimension of the semisimple part $d(B_i)$. Applying Kemer lemma 1 we have that $\alpha \geq \max_i(d(B_i))$. On the other hand if $\alpha > d(B_i)$, any multilinear polynomial with more than $n_{B_i} - 1$ alternating sets of cardinality α is in $id^H(B_i)$ (any alternating set must have at least one radical evaluation) and hence if $\alpha > \max_i(d(B_i))$, any polynomial as above is an identity of $B_1 \times \cdots \times B_q$ and hence of A . This contradicts the definition of the parameter α and hence $\alpha = \max_i(d(B_i))$. Now take an alternating set of cardinality $\alpha + 1$. In every such set we must have a radical evaluation or elements from different full algebras. If

they come from different full algebras we get zero. If we get a radical element then we cannot pass $n_A - 1$. \square

The next definition is key in the proof of Kemer's Lemma 2 (see below).

Definition 6.2. Notation as above. Let f be a multilinear polynomial which is not in $id^H(A)$. We say that A has property K with respect to f if f vanishes on any evaluation on A with less than $n_A - 1$ radical substitutions.

We say that a finite dimensional H -module algebra A has property K if it satisfies the property with respect to some nonidentity multilinear H -polynomial.

Proposition 6.3. *Let A be H -basic algebra. Then it has property K . Moreover there is a multilinear H -polynomial which satisfies property K and is full.*

Before proving the proposition we introduce a construction which will enable us to put some "control" on the nilpotency index of (the radical of) finite dimensional H -module algebras which are H -PI equivalent.

Let B be any finite dimensional H -module algebra and let $B' = \overline{B} * F^H \langle x_1, \dots, x_t \rangle$ be the co-product of the Free H -module algebra on the generators $\{x_1, \dots, x_t\}$ with the algebra \overline{B} , the semisimple component of B . We define an H action in the following fashion

$$h \cdot b_1 f_1 \cdots f_k b_{k+1} = h_{(1)}(b_1) h_{(2)}(f_1) \cdots h_{(2k)}(f_k) h_{(2k+1)}(b_{k+1})$$

where $b_1, \dots, b_k \in \overline{B}$ and $f_1, \dots, f_k \in F^H \langle x_1, \dots, x_t \rangle$. The number of variables we take is at least the dimension of $J(B)$. Let I_1 be the H -ideal of B' generated by all evaluations of polynomials of $id^H(B)$ on B' and let I_2 be the H -ideal generated by all variables x_i^h , where $h \in H$. Consider the H -module algebra $\widehat{B}_u = B' / (I_1 + I_2^u)$.

Proposition 6.4. *The following hold:*

- (1) $id^H(\widehat{B}_u) = id^H(B)$ whenever $u \geq n_B$ (n_B denotes the nilpotency index of $J(B)$).
In particular \widehat{B}_u and B have the same index.
- (2) \widehat{B}_u is finite dimensional.
- (3) The nilpotency index of $J(\widehat{B}_u)$ is $\leq u$.

Proof. Note that by the definition of \widehat{B}_u (modding B' by the ideal I_1), $id^H(\widehat{B}_u) \supseteq id^H(B)$. On the other hand there is a surjection $\phi : \widehat{B}_u \rightarrow B$ which maps the variables $\{x_i\}$ onto a spanning set of $J(B)$ and \overline{B} is mapped isomorphically. The ideal I_1 consist of all evaluation of $id^H(B)$ on B' and hence is contained in $ker(\phi)$. Also the ideal I_2^u is contained in $ker(\phi)$ since $u \geq n_B$ and $\phi(x) \in J$. This shows (1).

To see (2) observe that any element in \widehat{B}_u is represented by a sum of elements the form $b_1 z_1^{g_1} b_2 z_2^{g_2} \cdots b_j z_j^{g_j} b_{j+1}$ where $j < u$, $b_i \in \overline{B}$, $z_i \in \{x_1, \dots, x_t\}$ and g_i is in a basis of H . In order to prove the 3rd statement, note that I_2 generates a radical ideal \widehat{B}_u and since $B' / I_2 \cong \overline{B}$ we have that

$$\widehat{B}_u / I_2 \cong B' / (I_1 + I_2^u + I_2) = B' / (I_1 + I_2) \cong (B' / (I_2)) / I_1 = \overline{B} / I_1 = \overline{B}$$

(the last equality follows from the fact that $\overline{B} \subseteq B$). We therefore see that I_2 generates the radical in \widehat{B}_u , and hence its nilpotency index is bounded by u as claimed.

Proof. (of Proposition 6.3) Let B_1, \dots, B_q be the H -module algebras defined in lemma 5.10 and consider the H -module algebra $\widehat{A}_u = A'/(I_1 + I_2^u)$ (from the proposition above). It is clear that $id^H(\widehat{A}_{n_A-1} \times B_1 \times \dots \times B_q) \supsetneq id^H(A)$. We show that if the proposition is false (for A), then there is an equality. Since $\text{Par}(B_i), \text{Par}(\widehat{A}_{n_A-1}) < \text{Par}(A)$ we get a contradiction.

Take a multilinear polynomial $f = f(x_1, \dots, x_n)$ which is not in $id^H(A)$ and consider a non zero evaluation $\bar{x}_1, \dots, \bar{x}_n$ on A . Suppose that $\bar{x}_1, \dots, \bar{x}_v \in J$ and the rest are in \overline{A} . If $v < n_A$, then $f' = f(x_1, \dots, x_s, \bar{x}_{v+1}, \dots, \bar{x}_n) \in B'$ and is non zero in \widehat{A}_{n_A-1} , since otherwise $f' = \sum_i g_i(x_1, \dots, x_v, Y)$, where $g_i \in id^H(A)$ and $Y \subseteq \overline{A}$. So by substituting \bar{x}_i instead x_i , we will get that $f(\bar{x}_1, \dots, \bar{x}_n) = 0$. If not all the simple components of A appear in $\bar{x}_{v+1}, \dots, \bar{x}_n$ we get (see the proof of lemma 5.10) that f is a non identity of one of the B_i . By our assumption these are the only options. Hence, in any case f is a non identity of the product $\widehat{A}_{n_A-1} \times B_1 \times \dots \times B_q$ as claimed. \square

\square

In the next lemma we deal with properties which are preserved in H - T -ideals.

Lemma 6.5. *Let A be H -basic. The following hold.*

- (1) *Let $f \notin id^H(A)$ be a multilinear polynomial and suppose A is full with respect to nonzero evaluations of f on A , that is, in any nonzero evaluation of f on A we must have semisimple values from all H -simple components. Then if $f' \in \langle f \rangle_H$ is multilinear ($\langle f \rangle_H = H$ - T -ideal generated by f) is a nonidentity of A then it is full with respect to any nonzero evaluation on A .*
- (2) *Let $f \notin id^H(A)$ be multilinear and suppose it is μ -fold alternating on disjoint sets of cardinality $d(A) = \dim_F(\overline{A})$. If $f' \in \langle f \rangle_H$ is a nonidentity of A , then there exists a nonidentity $f'' \in \langle f' \rangle_H$ of A , which is multilinear and μ -fold alternating on sets of cardinality $d(A)$. In other words, the property of being μ -fold alternating on sets of cardinality $d(A)$ is A -Phoenix.*
- (3) *Property K is strictly A -Phoenix.*

Proof. Suppose $f(x_1, \dots, x_n)$ is a multilinear polynomial which satisfies the condition in 1. It is sufficient to show the condition remains valid if f' is multilinear and has the form (a) $f' = \sum_i p_i \cdot f \cdot q_i$ (b) $f'(z_1, \dots, z_t, x_2, \dots, x_n) = f(Z, x_2, \dots, x_n)$ where $Z = z_1^{h_1} \dots z_t^{h_t}$ is a multilinear monomial consisting of variables disjoint to the variables of $f(x_1, \dots, x_n)$. If $f' = \sum_i p_i \cdot f \cdot q_i$ then any nonzero evaluation of f' arises from a nonzero evaluation of f and so the claim is clear in this case. Let $f'(z_1, \dots, z_t, x_2, \dots, x_n) = f(Z, x_2, \dots, x_n)$ and suppose $x_i = \hat{x}_i$ and $z_i = \hat{z}_i$ is a non vanishing evaluation of f' . If an H -simple component A_1 say, is not represented, then the same simple component is not represented in the evaluation $x_1 = \hat{z}_1^{h_1} \dots \hat{z}_t^{h_t}, x_2 = \hat{x}_2, \dots, x_n = \hat{x}_n$ and hence f vanishes. We see that f' vanishes on any evaluation which misses a simple component.

For the second part of the lemma note that if f is multilinear and has μ -folds of alternating sets of cardinality $d(A) = \dim_F(\overline{A})$ then clearly it vanishes on any evaluation unless it visits in all simple components and hence the result follows from the first part of the Lemma and Kemer Lemma 1.

We now turn to the proof of the 3rd part of the lemma. If $f' = \sum_i g_i f p_i$ then it is clear that if an evaluation of f' has less than $n_A - 1$ radical evaluations then with that evaluation f has less than $n_A - 1$ radical evaluations and hence vanishes. This implies the vanishing of f' . If an evaluation of $f'(z_1, \dots, z_t, x_2, \dots, x_n) = f(Z, x_2, \dots, x_n)$ has less than $n_A - 1$ radicals, then this corresponds to an evaluation of $f(x_1, \dots, x_n)$ with less than $n_A - 1$ radicals and hence vanishes. \square

We can now state and prove Kemer's lemma 2.

Lemma 6.6 (Kemer's lemma 2). *Let A be H -full algebra. Suppose $\text{Par}(A) = (d = d(A), n_A - 1)$. Then for any integer ν there exists a multilinear nonidentity f with μ -alternating sets of cardinality d (small sets) and precisely $n_A - 1$ alternating sets of variables of cardinality $d + 1$ (big sets).*

Remark 6.7. The theorem is clear either in case A is radical or semisimple (i.e. H -simple). Hence for the proof we assume that $q \geq 1$ (the number of simple components of A) and $n_A > 1$.

Remark 6.8. Any nonzero evaluation of such f must consists only of semisimple evaluations in the ν -folds and each one of the big sets (namely the sets of cardinality $d + 1$) must have exactly one radical evaluation.

Proof. By the preceding Lemma we take a multilinear nonidentity H -polynomial f , with respect to which A is full and has property K . Let us fix a nonzero evaluation $x \mapsto \hat{x}$ realizing the “full” property. Note (by the remark above) that by the construction of f , being the evaluation nonzero, precisely $n_A - 1$ variables must obtain radical values and the rest of the variables obtain semisimple values. Let us denote by w_1, \dots, w_{n_A-1} the variables that obtain radical values (in the evaluation above) and by $\hat{w}_1, \dots, \hat{w}_{n_A-1}$ their corresponding values. By abuse of language we refer to the variables w_1, \dots, w_{n_A-1} as *radical* variables.

We will consider four cases. These correspond to whether A has or does not have an identity element and whether q (the number of H -simple components) > 1 or $q = 1$.

Case (1, 1) (A has an identity element and $q > 1$).

By linearity we may assume the evaluation of any radical variable w_i is of the form $1_{A_{j(i)}} \hat{w}_i 1_{A_{\tilde{j}(i)}}$, $i = 1, \dots, n_A - 1$, where 1_{A_k} is the identity element of the H -simple component A_k . Note that the evaluation remains full (i.e. visits any simple component of A).

Choose a monomial X of f which does not vanish upon the above evaluation. Notice that the variables of X which get semisimple evaluations from *different* H -simple components must be separated by radical variables.

Consider the radical evaluations which are bordered by pairs of elements $(1_{A_{j(i)}}, 1_{A_{\tilde{j}(i)}})$ where $j(i) \neq \tilde{j}(i)$ (i.e. belong to different H -simple components). Then it is clear that every simple component is represented by one of the elements in these pairs.

For $t = 1, \dots, q$ we fix a variable w_{r_t} whose radical value is $1_{A_{j(r_t)}} \hat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$ where

- (1) $j(r_t) \neq \tilde{j}(r_t)$ (i.e. different H -simple components).
- (2) One of the element $1_{A_{j(r_t)}}, 1_{A_{\tilde{j}(r_t)}}$ is the identity element of the t -th simple component.

We refer to that element as the *idempotent attached* to the simple component A_t .

Remark. Note that we may have $w_{r_t} = w_{r_{t'}}$ even if $t \neq t'$.

Next replace the variables w_{r_t} , $t = 1, \dots, q$ by $z_{r_t} y_{r_t} z'_{r_t} w_{r_t}$ or $w_{r_t} z_{r_t} y_{r_t} z'_{r_t}$ (and obtain a new polynomial f_1) according to the location of the primitive H -invariant idempotent attached to the t -th simple component. Clearly, by evaluating the variables y_{r_t} , z_{r_t} and z'_{r_t} by $1_{A_{j(r_t)}}$ (or $1_{A_{\tilde{j}(r_t)}}$) the value of f_1 remains the same as f_1 under the original substitution and in particular nonzero. For later reference we call the variables z_{r_t} and z'_{r_t} *frame variables* and consider the evaluation $1_{A_{j(r_t)}} \rightarrow z_{r_t}, z'_{r_t}$ (or $1_{A_{\tilde{j}(r_t)}}$).

Applying lemma 5.12 we can replace (in f_1) the variable y_{r_t} , $t = 1, \dots, q$, by a μ -fold alternating polynomial (on the distinct sets U_l^t) $Z_{r_t} = Z_{r_t}(U_1^t, \dots, U_\nu^t; Y_t)$, and obtain a nonzero polynomial f_2 . Here, the sets U_l^t , $l = 1, \dots, \nu$ are each of cardinality $\dim_F(A_t)$. Now, if we further alternate the sets U_l^1, \dots, U_l^q for $l = 1, \dots, \nu$ together (that is for each l , apply $\text{Alt}_{U_l^1 \cup \dots \cup U_l^q}$) we obtain a nonidentity polynomial with ν -folds of (small) sets of alternating variables where each set is of cardinality $\dim(\bar{A})$. In the sequel we fix an evaluation of the polynomials Z_{r_t} (or \tilde{Z}_{r_t}) so the entire polynomial obtains a nonzero value.

Our next task is to construct such polynomial with an extra $n_A - 1$ alternating sets of cardinality $d + 1$ (big sets). Consider the radical variables w_{r_t} , $t = 1, \dots, q$ with radical evaluations $1_{A_{j(r_t)}} \hat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$, $j(r_t) \neq \tilde{j}(r_t)$ (i.e. different H -simple components).

We attach each variable w_{r_t} to one alternating set U_l^1, \dots, U_l^q (some l). We see that any nontrivial permutation of w_{r_t} with one of the variables of U_l^1, \dots, U_l^q , keeping the evaluation above, will yield a *zero value* since the primitive H -invariant idempotents values in frames variables of each Z_{r_1}, \dots, Z_{r_q} belong to the same H -simple components whereas the pair of idempotents in $1_{A_{j(r_t)}} \hat{w}_{r_t} 1_{A_{\tilde{j}(r_t)}}$ belong to different H -simple components. Thus we may alternate the variable w_{r_t} with $U_{l_t}^1, \dots, U_{l_t}^q$, $t = 1, \dots, q$ and obtain a multilinear nonidentity of A . Next we proceed in a similar way with any remaining variable w_i whose evaluation is $1_{A_{j(i)}} \hat{w}_i 1_{A_{\tilde{j}(i)}}$ and $j(i) \neq \tilde{j}(i)$.

Finally we need to attach the radical variables w_i whose evaluation is $1_{A_{j(i)}} \hat{w}_i 1_{A_{\tilde{j}(i)}}$ where $j(i) = \tilde{j}(i)$ (i.e. the same simple component) to some small sets. We claim also here that if we attach the variable w_i to the sets U_l^1, \dots, U_l^q (some l), any nontrivial permutation yields a zero value, and hence the value of the entire polynomial remains unchanged. If we permute w_i with an element $u_0 \in U_l^k$ which is bordered by idempotents different from $1_{A_{j(i)}}$ we obtain zero. On the other we claim that any permutation of w_i with an element $u_0 \in U_l^k$ which is bordered by the idempotent $1_{A_{j(i)}}$ corresponds to an evaluation of the original polynomial with fewer radical values and then we will be done by the property K . In order to simplify our notation let $\{U_l^1, \dots, U_l^q\} = \{U^1, \dots, U^q\}$ (omit the index l) and suppose without loss of generality, that $u_0 \in U^1$. Permuting the variables w_i and u_0 (with their corresponding evaluations) we see that the polynomial $Z_{r_1} = Z_{r_1}(U^1 = U_1^1, \dots, U_\nu^1; Y_1)$ (or \tilde{Z}_{r_1}) with w_i replacing u_0 , obtains a radical value which we denote by $\hat{\hat{w}}$. Returning to our original polynomial f , we obtain the same value if we evaluate the variable w_i by a suitable semisimple element, the variable w_{r_1}

by $\widehat{w}\widehat{w}_{r_1}$ (or $\widehat{w}_{r_1}\widehat{w}$) and the evaluation of any semisimple variable remains semisimple. It follows that if we make such a permutation for a unique radical variable w_i , the value amounts to an evaluation of the original polynomial with $n_A - 2$ radical evaluations and hence vanishes. Clearly, composing $p > 0$ permutations of that kind yields a value which may be obtained by the original polynomial f with $n_A - 1 - p$ radical evaluations and hence vanishes by property K . This completes the proof of the lemma where A has identity and q , the number of simple components, is > 1 .

Case (2, 1). Suppose now A has no identity element and $q > 1$. Let $A_0 = A \oplus F1$, where $h(1) = \epsilon(h)1$. The proof in this case is basically the same as in the case where A has an identity element. Let $e_0 = 1 - 1_{A_1} - 1_{A_2} - \cdots - 1_{A_q} \in A_0$ and attach e_0 to the set of elements which border the radical values \widehat{w}_j . A similar argument shows that also here every H -simple component (A_1, \dots, A_q) is represented in one of the bordering pairs where the partners are *different* (the point is that one of the partners (among these pairs) may be e_0). Now we complete the proof exactly as in case (1, 1).

Case (2, 2). In order to complete the proof of the lemma we consider the case where A has no identity element and $q = 1$. The argument in this case is different. For simplicity we denote by $e_1 = 1_{A_1}$ and $e_0 = 1 - e_1$. Let $f(x_1, \dots, x_n)$ be a nonidentity of A which satisfies property K and let $f(\widehat{x}_1, \dots, \widehat{x}_n)$ be a nonzero evaluation for which A is full. If $e_1 f(\widehat{x}_1, \dots, \widehat{x}_n) \neq 0$ (or $f(\widehat{x}_1, \dots, \widehat{x}_n) e_1$) we proceed as in case (1, 2). To treat the remaining case we may assume further that

$$e_0 f(\widehat{x}_1, \dots, \widehat{x}_n) e_0 \neq 0$$

First note, by linearity, that each one of the radical values \widehat{w} may be bordered by one of the pairs $\{(e_0, e_0), (e_0, e_1), (e_1, e_0), (e_1, e_1)\}$ so that if we replace the evaluation \widehat{w} (of w) by the corresponding element $e_i \widehat{w} e_j$, $i, j = 0, 1$, we get nonzero.

Now, one of the radical values (say \widehat{w}_0) in $f(\widehat{x}_1, \dots, \widehat{x}_n)$ allows a bordering by the pair (e_0, e_1) (or (e_1, e_0)), then replacing w_0 by $w_0 y$ (or $y w_0$) yields a nonidentity. Invoking Lemma 5.12 we may replace the variable y by a polynomial p with μ -folds of alternating (small) sets of cardinality $\dim_F(\overline{A}) = \dim_F(A_1)$. Then we attach the radical variable w_0 to one of the small sets. Clearly, the value of any alternation of this (big) set is zero since the borderings are different. The remaining possible values of radical variables are either $e_0 \widehat{w} e_0$ or $e_1 \widehat{w} e_1$. Note that since semisimple values can be bordered only by the pair (e_1, e_1) , any alternation of the radical variables whose radical value is $e_0 \widehat{w} e_0$ with elements of a small set vanishes and again the value of the polynomial remains unchanged. Finally we attach the remaining radical variables (whose values are to suitable small sets in p . Here, any alternation vanishes because of property K . This settles this case. Obviously, the same holds if the bordering pair above is (e_1, e_0) . \square

Corollary 6.9. *If A is basic then its H -Kemer index (α, s) equals $(d, n_A - 1)$.*

Corollary 6.10. *Let A be a finite dimensional H -module algebra, then there is a number μ'_A such that every H -Kemer polynomial f of A of rank at least μ'_A satisfies the A -Phoenix property.*

Proof. Suppose A is H -basic. Clearly if f is H -Kemer of rank $\mu \geq \mu_A = \mu'_A$ then A is full and satisfies property K with respect to f . The Corollary now follows from Lemmas lemma 6.5 and lemma 6.6.

Consider now the general case, we may suppose that $A = B_1 \times \cdots \times B_q$ is a product of H -basic algebras. Let f' be a multilinear consequence of f which is not an identity of A . Thus f' must be a non identity of (at least) one of the B_i , say B_1 . Therefore, f is also a non identity of B_1 . Thus, if $\mu \geq \mu_{B_1}$ we can conclude that $\text{Ind}(A) \leq \text{Ind}(B_1)$, so $\text{Ind}(A) = \text{Ind}(B_1)$. Hence f is H -Kemer of B_1 . By the previous paragraph we are done if we set $\mu'_A = \max\{\mu_{B_1}, \dots, \mu_{B_q}\}$. \square

7. TECHNICAL TOOLS

7.1. Affine relatively H -module algebras . Recall that an algebra W satisfies the t th Capelli identity if any multilinear polynomial having an alternating set of cardinality (at least) t is an H -identity of W . The purpose of this section is to prove that for any such algebra one can assume that the corresponding relatively free algebra \mathcal{W} is generated by (only) $t - 1$ variables. More precisely, we will show that if

$$\mathcal{W} = F^H \langle x_1, \dots, x_{t-1} \rangle / id^H(W) \cap F^m \langle x_1, \dots, x_{t-1} \rangle$$

then $id^H(W) = id^H(\mathcal{W})$. To this end we recall some basic results (and fix notation) from the representation theory of S_n (the symmetric group on n elements) and their application to PI theory.

Let $P_n^H(W) = P_n^H / (P_n \cap id^H(W))$, where P_n^H is the space (of dimension $(\dim_F H)^n \cdot n!$) of all multilinear polynomials with variables x_1, \dots, x_n . The group S_n acts on (right action!) $P_n^H(W)$ via $\sigma \cdot \overline{x_{i_1}^{h_1} \cdots x_{i_n}^{h_n}} = \overline{x_{\sigma(i_1)}^{h_1} \cdots x_{\sigma(i_n)}^{h_n}}$ and hence we may consider its decomposition into irreducible submodules. By the representation theory of S_n in characteristic zero, any such submodule can be written as $FS_n e_{T_\mu} \cdot f$, where f is some polynomial in $P_n^H(W)$, T_μ is some Young tableau of the partition μ (of n) and

$$e_{T_\mu} = \sum_{\sigma \in \mathcal{R}_{T_\mu}, \tau \in \mathcal{C}_{T_\mu}} (-1)^\tau \sigma \tau.$$

(here \mathcal{R}_{T_μ} and \mathcal{C}_{T_μ} are the rows and columns stabilizers respectively). Clearly, if $f \in P_n^H(W)$ is nonzero, then there is some partition μ and a (standard) tableau T_μ such that $e_{T_\mu} \cdot f$ is nonzero.

We are ready to prove the main result of this section.

Theorem 7.1. *Let W be an H -module algebra which satisfies the t th Capelli identity. Then $id^H(\mathcal{W}) = id^H(W)$ where \mathcal{W} is the relatively free H -module algebra of W generated by $t - 1$ variables.*

Proof. It is clear that $id^H(W) \subset id^H(\mathcal{W})$. For the other direction suppose f is a multilinear nonidentity of W of degree n . Then, by the theorem above, there is a partition μ of n and a tableau T_μ such that $g = e_{T_\mu} \cdot f$ is a nonidentity of W .

Let $g_0 = \sum_{\tau \in \mathcal{C}_{T_\mu}} (-1)^\tau \tau \cdot f = \sum_{\tau \in \mathcal{C}_{T_\mu}(1)} (-1)^\tau \tau \cdot \left(\sum_{k=1}^l (-1)^{\tau_k} \tau_k \cdot f \right)$, where $\mathcal{C}_{T_\mu}(1)$ is the stabilizer of the first column of T_μ and τ_1, \dots, τ_l is a full set of representatives of $\mathcal{C}_{T_\mu}(1)$ -cosets in \mathcal{C}_{T_μ} .

Let $h(\mu)$ (the height of μ) denote the number of rows in T_μ . If $h(\mu) \geq t$, the polynomial g_0 is alternating on the variables of the first column and hence by assumption is an identity of W . But in that case also the polynomial $g = \sum_{\sigma \in R_{T_\mu}} \sigma \cdot g_0$ is in $id^H(W)$ contradicting our assumption and so $h(\mu)$ must be smaller than t .

Since $g = \sum_{\sigma \in R_{T_\mu}} \sigma \cdot g_0$, it is symmetric in the variables corresponding to any row of T_μ and so if for any $i = 1, \dots, h(\mu)$ we replace by y_i all variables in g corresponding to the i th row we obtain a polynomial \hat{g} which yields g by multilinearization. In particular $g \in id^H(W)$ if and only if $\hat{g} \in id^H(W)$. Finally, \hat{g} can be regarded as an element of \mathcal{W} (at most $t - 1$ variables) and nonzero, thus g is a nonidentity of \mathcal{W} and hence also f . \square

Remark 7.2. In the sequel, if W satisfies the t th Capelli identity, we'll consider affine relatively free H -module algebras \mathcal{W} with at least $t - 1$ generating variables.

Definition 7.3. Suppose W is an affine H -module algebra. Any algebra of the form

$$F^H \langle x_1, \dots, x_t \rangle / id^H(W) \cap F^H \langle x_1, \dots, x_t \rangle$$

having the same T ideal as W is called *affine relatively free H -module algebra of W* .

We close this subsection with the following useful lemma.

Corollary 7.4. *Suppose \mathcal{W} is an relatively free H -module algebra of W (in particular we will be interested in the case where \mathcal{W} is affine). Let I be any H - T ideal and denote by \hat{I} the ideal of \mathcal{W} generated (or consisting rather) by all evaluation on \mathcal{W} of elements of I . Then $id^H(\mathcal{W}/\hat{I}) = id^H(W) + I$.*

7.2. Shirshov base.

Definition 7.5. Let W be an affine algebra over F . Let a_1, \dots, a_s be a generating set of W . Let t be a positive integer and let Y be the set of words in a_1, \dots, a_s of length $\leq t$. We say that W has a Shirshov base of length t and of height h if W is spanned (over F) by elements of the form $y_1^{n_1} \cdots y_l^{n_l}$, where $y_i \in Y$ and $l \leq h$.

The following fundamental theorem was proved by Shirshov.

Theorem 7.6. *If an affine algebra W has a multilinear PI of degree t , then it has a Shirshov base of length t and some height h where h depends only on t and the number of generators of W .*

In fact, there is an important special case where we can get even “closer” to representability.

Theorem 7.7. *Let C be a commutative algebra over F and let $W = C \langle a_1, \dots, a_s \rangle$. Suppose W has a Shirshov base. If for every $i = 1, \dots, s$, the element a_i is integral over C , then W is a finite module over C .*

If in addition, our commutative algebra C is Noetherian and unital we reach our goal, as the next theorem shows.

Theorem 7.8 (Beidar [5]). *Let W be an algebra and C be a unital commutative Noetherian F -algebra. If W is a finite module over C , then W is representable.*

Following the proof in [2] (Theorem 1.6.22), it is easy to generalize this theorem to H -module algebras:

Theorem 7.9. *Let W be an H -module C -algebra, where C is a unital commutative Noetherian F -algebra (so in particular $h \cdot (cw) = c(h \cdot w)$ for $c \in C, h \in H$ and $w \in W$). If W is a finite module over C , then W is H -representable (i.e. there is a field extension K of F and an H -module K -algebra A , which is finite dimensional over K , such that W is H -module F -subalgebra of A).*

Proof. If W does not posses an identity element we may replace W by $W \oplus C1$ (H acts on $C1$ by $h \cdot c1 = c\epsilon(h)1$). Moreover, the map $\pi : C \rightarrow Z(W)$ given by $c \rightarrow c1$ is a homomorphism. The image of π is commutative unital Noetherian F -algebra. Thus we may also assume that C is embedded in the center of W .

Next, if I and J are zero intersecting H -ideals of W , we have $W \hookrightarrow W/I \times W/J$ is an H -module C -algebras embedding. By Noetherian induction for H -ideals, we obtain that W is H -embedded in a finite product of Noetherian H -module C -algebras each having no zero intersecting H -ideals (H -irreducible) apart the zero ideals. Therefore, we may also assume W is H -irreducible.

Suppose $z \in C$ is non nilpotent. Since W is Noetherian there is some k for whcih $\text{ann}_W(z^k) = \text{ann}_W(z^{k+1}) = \dots$. Hence, $\text{ann}_W(z^k) \cap z^k W = 0$ (indeed, if $x = z^k w \in \text{ann}_W(z^k)$, then $z^{2k} w = 0 \Rightarrow w \in \text{ann}_W(z^{2k}) = \text{ann}_W(z^k) \Rightarrow x = 0$). Since $\text{ann}_W(z^k)$ and zW are H -ideals (recall that $h \cdot (zw) = z(h \cdot w)$) and W is an H -irreducible, we must conclude $\text{ann}_W(z^k) = 0$. In other words, z is not zero divisor in W .

Denote by S all the non-nilpotent elements of C . By the previous paragraph, W H -embeds into $W_1 = S^{-1}W$ (the H -action is given by $h \cdot (s^{-1}w) = s^{-1}h(w)$). $C_1 = S^{-1}C$ is Noetherian and local (see Lemma 1.6.27 in [2]) with $J(C)$ equals to a nilpotent maximal ideal. Hence, by Lemma 16.25 in [2] C contains a field K with the property $K \simeq C/J(C)$.

Denote by k the nilpotency index of $J(C)$. So we have

$$J(C) \supseteq J(C)^2 \supseteq \dots \supseteq J(C)^{k-1}.$$

Since $J(C)^i/J(C)^{i+1}$ is finite over C (since C is Noetherian), it is also finite over $C/J(C) = K$. Hence C is finite over K . The theorem follows because W_1 is finite over C . \square

8. RELATIVELY FREE H -MODULE ALGEBRA OF A FINITE DIMENSIONAL H -MODULE ALGEBRA

Suppose A is a finite dimensional H -module F -algebra and \mathcal{A} is its corresponding affine relatively free H -module algebra which is H -generated by the variables x_1, \dots, x_t . Suppose further that a_1, \dots, a_l is an F -basis for A and consider the map $\phi : \mathcal{A} \rightarrow A \otimes_F K$, where $K = F(\{\lambda_{i,j} \mid i = 1, \dots, t; j = 1 \dots l\})$, induced by

$$x_i \rightarrow \sum_{k=1}^l \lambda_{i,k} a_k.$$

It is easy to check that if we denote by \mathcal{A}' the image of this H -map we will obtain that \mathcal{A} and \mathcal{A}' are H -isomorphic. Therefore, we will abuse notation and denote also the image by \mathcal{A} .

Suppose now that $A = A_1 \times \cdots \times A_s$ is a product of H -basic algebras. Denote by R_1, \dots, R_s the H -invariant semisimple part of A_1, \dots, A_s respectively. We may embed R_i into $\text{End}_F(R_i)$ (F -algebras embedding) and define

$$\text{tr} : R_1 \times \cdots \times R_s \rightarrow F^{\times s}$$

by

$$\text{tr}(a_1, \dots, a_s) = (\text{tr}_{\text{End}_F(R_1)}(a_1), \dots, \text{tr}_{\text{End}_F(R_s)}(a_s)).$$

Furthermore, tr can be extended to a function $A \rightarrow F^{\times s}$ by declaring that the trace of a radical element is $(0, \dots, 0)$. Since $F^{\times s}$ embeds into $R_1 \times \cdots \times R_s$, it acts on A . Finally, notice that each semisimple $a \in A$ satisfies a Cayley-Hamilton identity of degree $d = \max\{d_1 = \dim_F R_1, \dots, d_s = \dim_F R_s\}$.

Lemma 8.1. *Suppose $f(x_1, \dots, x_d, Y)$ is an H -Kemer polynomial of rank at least $\mu_A + 1$ and $\{x_1, \dots, x_d\}$ is one of the small sets, then:*

$$\text{tr}(a_0)f(a_1, \dots, a_d, \bar{Y}) = \sum_{k=1}^d f(a_1, \dots, a_{k-1}, a_0 a_k, a_{k+1}, \dots, a_d, \bar{Y}),$$

where $a_0, \dots, a_d \in A$ and \bar{Y} is some evaluation of the variables of Y by elements of A .

Proof. See Proposition 10.5 in [4]. □

Corollary 8.2. *If I is an ideal of \mathcal{A} generated (as an H - T -ideal) by d -alternating H -polynomials, then $\text{tr}(x_0) \cdot f \in I$, where $x_0 \in \mathcal{A}$ and $f \in I$.*

9. Γ -PHOENIX PROPERTY

Suppose Γ is an H - T -ideal containing a Capelli identity. We know this implies that Γ contains the H - T -ideal of a finite dimensional H -module F -algebra A . If we denote by p_Γ and p_A the H -Kemer index of Γ and A respectively, then $p_\Gamma \leq p_A$. Our goal in this section is to show that it is possible to replace A by another finite dimensional H -module algebra B which is “closer” to Γ in the sense that its H -Kemer index and H -Kemer polynomials are exactly as those of Γ . This will allow us to deduce the Phoenix property for H -Kemer polynomials of Γ from (the already established) Phoenix property for H -Kemer polynomials of B .

Let A be a finite dimensional H -module algebra which is a direct product of basic algebras $A_1 \times \cdots \times A_s$. Let p_A and p_i denote the H -Kemer index of A and A_i , $i = 1, \dots, s$ respectively. We let $\mu_i = \mu_{A_i}$ and write μ_0 for the maximum of $\{\mu_1, \dots, \mu_s\}$.

Proposition 9.1. *Let Γ and A as above. Then there exist a representable H -module algebra B with the following properties:*

- (1) $\text{id}^H(B) \subseteq \Gamma$.
- (2) The Kemer index p_B of B coincides with p_Γ .
- (3) Γ and B have the same H -Kemer polynomials corresponding to every μ which is $\geq \mu_0$.

Any A satisfying (2) and (3) is called H -Kemer equivalent to W .

Corollary 9.2. *By extending scalars to a larger field we may assume the H -module algebra B is finite dimensional over F .*

of 9.1. Let \mathcal{B} be a Shirshov base of \mathcal{A} . Consider the constructions in section 8 so that \mathcal{A} is an H -module F -subalgebra of $A \otimes_F K$. Denote by C the unital F -subalgebra of $K^{\times s}$ generated by the characteristic values of the elements of \mathcal{B} . Notice that this is a Noetherian F -algebra. Finally, define the H -module C -algebra $\mathcal{A}_C = C \cdot \mathcal{A}$.

Let \mathcal{I} be the set of all evaluations in \mathcal{A} of all H -Kemer polynomials of A which are inside Γ . It is clear \mathcal{I} is an H -ideal of \mathcal{A} . By theorem 7.9 and theorem 7.7 we know that $\mathcal{A}_C/C\mathcal{I}$ is representable. So, since 8.2 implies $\mathcal{A}/\mathcal{I} \subseteq \mathcal{A}_C/C\mathcal{I}$ we conclude that \mathcal{A}/\mathcal{I} is representable.

Furthermore, $id^H(\mathcal{A}/\mathcal{I}) \subseteq \Gamma$ and $\text{Ind}(\mathcal{A}_C/C\mathcal{I}) < \text{Ind}(A)$. So by extending the field F we are allowed to assume \mathcal{A}/\mathcal{I} is a finite dimensional H -module algebra. By induction on the H -Kemer index we obtain a finite dimensional (over some extension field of F) H -module algebra which satisfies (1) and (2). In order to get also (3), we repeat the process above one final time. \square

Corollary 9.3 (Phoenix property). *Let A be an affine H -module algebra, then there is a number μ'_W such that every H -Kemer polynomial f of A of degree at least μ'_W satisfies the W -Phoenix property.*

Proof. By the previous theorem we may switch W by a finite dimensional H -module algebra without changing the H -Kemer index and polynomials. So the corollary follows from 6.10. \square

Definition 9.4. Let W be an affine H -module algebra and let B be a finite dimensional algebra as in 9.1. Denote by ν_W the number $\max\{\mu_0, \mu_W, \mu_B, \mu'_W, \mu'_B\}$. Informally, for $\mu \geq \nu_W$ all the theorem concerning H -Kemer polynomials of W are true.

10. REPRESENTABLE SPACES

In this section we show the existence of a representable algebra B_Γ satisfying the properties:

- $id^H(B_\Gamma) \supseteq \Gamma$.
- All H -Kemer polynomials of Γ are non-identities of B_Γ .

We have seen in section §9 that W is H -Kemer equivalent to a product of H -basic algebras $A = A_1 \times \cdots \times A_t$. Furthermore, theorem 7.1 says that there is a number l such that the relatively free H -module algebra of A on the set $\Sigma = \{y_1, \dots, y_l\}$ variables has the same H -identities as A . Denote this algebra by \mathcal{A} .

As before we identify \mathcal{A} with an H -module subalgebra of $A(\Lambda)$, where

$$\Lambda = \{\lambda_{k,i} | k = 1 \dots \dim R, i = 1 \dots l\}.$$

As in section section §9 we view $R(\Lambda)$ as a subalgebra of $\text{End}_K(R_1(\Lambda)) \times \cdots \times \text{End}_K(R_m(\Lambda))$, where $K = F(\Lambda)$, and consider the trace function $tr(a) = (tr(a), \dots, tr(a)) \in K^{\times m}$, where a is taken from $R(\Lambda)$. We may extend tr to $A(\Lambda)$ by declaring the trace of a nilpotent element is zero.

Next, consider a Shirshov base \mathcal{B} of \mathcal{A} which corresponds to the generators x^h (here $x \in \Sigma$ and h varies over a basis of H). This allows us to define the commutative unital F -algebra C generated by the characteristic values of elements of B (notice that each element in $A(\Lambda)$ satisfies a Cayley-Hamilton polynomial of degree $d = \alpha(W)$). It is clear that C is Noetherian which acts on $R(\Lambda)$.

Definition 10.1. Let R be an H -module algebra. Denote by $id_R^H(A)$ the H -ideal of R consisting of all evaluations of polynomials in $id^H(A)$ on R .

Denote by (d, s) the H -Kemer index of W . Consider the relatively free H -module algebra of A on the set of variables $\Sigma \cup X$, where $X = \cup_{i=1}^{\mu+1+r} X_i$, where $|X_1| = \dots = |X_{\mu+1}| = d$ and $|X_{\mu+2}| = \dots = |X_{\mu+r+1}| = d+1$ and μ is big enough so that $|X| \geq \dim J(A)$ and $\mu \geq \nu_W$.

Recall the construction from (the proof of) 6.3, which is in our case:

$$\mathcal{A}_2 = \frac{A_0 = C \cdot \mathcal{A} *_C C^H \{X\}}{id_{A_0}^H(A) + \langle X \rangle_H^{|X|}}.$$

We also define \mathcal{A}_1 to be the H -module algebra H -generated by \mathcal{A} and X . So:

$$\mathcal{A}_1 = \frac{A'_0 = F^H \{\Sigma \cup X\}}{id_{A'_0}^H(A) + \langle X \rangle_H^{|X|}}.$$

It is easy to see that $id^H(\mathcal{A}_1) = id^H(\mathcal{A}) = id^H(A)$. Note that this construction insures that any H -quotient of \mathcal{A}_2 is representable.

Definition 10.2. Let f be an H -Kemer polynomial of W with at least $\mu+1$ small sets. An evaluation of f on \mathcal{A}_1 is admissible if the following hold:

- (1) Precisely $\mu+1$ small sets in f , say $\dot{X}_1, \dots, \dot{X}_{\mu+1}$, are evaluated bijectively on the sets $X_1, \dots, X_{\mu+1}$.
- (2) All big sets of f are evaluated bijectively on the sets $X_{\mu+2}, \dots, X_{\mu+1+r}$.
- (3) The rest of the variables in f are evaluated on $H \cdot \pm$.

Denote by \mathcal{S} the **set** of all admissible evaluations of all H -Kemer polynomials of W .

Our goal is to prove that \mathcal{S} projects injectively into $\mathcal{A}_3 = \mathcal{A}_2 / id_{\mathcal{A}_2}^H(W)$. After this is established it will be clear that:

- (1) \mathcal{A}_3 is representable.
- (2) $id^H(W) \subseteq id^H(\mathcal{A}_3)$.
- (3) W and \mathcal{A}_3 share the same H -Kemer polynomials.

Lemma 10.3. $id_{\mathcal{A}_2}^H(W) \cap \mathcal{S} = \{0\}$.

Proof. Suppose $f \in \mathcal{S}$ is also in $id_{\mathcal{A}_2}^H(W)$. So there are $g_i \in C$ and evaluations p_i of multilinear polynomials in $id^H(W)$ by elements from the set $X \cup \Sigma$ such that

$$f = \sum g_i p_i.$$

By specializing different $x \in X$ to 0 we may assume that each monomial of each p_i has at least one appearance of every $x \in X$. Since $\langle X \rangle_H^{|X|} = 0$, each p_i is multilinear in X .

It is easy to check that $\text{Alt}_{X_1}(f)$ is well defined on \mathcal{A}_1 . So

$$d! \cdot f = \sum g_i \text{Alt}_{X_1}(p_i).$$

Therefore, in $A_0/id_{A_0}^H(A)$ we get the equality:

$$d! \cdot f = \sum g_i \text{Alt}_{X_1}(p_i) + b,$$

where $b \in \langle X \rangle_H^{|X|}$. We may substitute instead of each $x \in X$ an element of the form $\bar{x} = \sum_{k=1}^{\dim_F A} a_k \tau_{k,x}$, where $\tau_{k,x}$ is a commutative indeterminate and $a_1, \dots, a_{\dim A}$ is a basis of A . By lemma 8.1, $\overline{g_i \text{Alt}_{X_1}(p_i)}$ is equal to $\overline{\psi_i}$, where $\psi_i \in F^H \{X \cup \Sigma\}$ (multilinear in X) is in $id^H(W)$. Thus,

$$d! \cdot f \equiv_{\text{mod } id^H(A)} \sum \psi_i \in id^H(W).$$

Since $id^H(A) \subseteq id^H(W)$, we got a contradiction to f being an H -Kemer polynomial of W . \square

Corollary 10.4. *Let f be any H -Kemer polynomial of the H -module algebra W (at least $\mu + 1$ small sets). Then $f \notin id^H(\mathcal{A}_3)$.*

11. FINALIZATION OF THEOREM 1.1

We have all the ingredients needed to prove the main theorem.

Proof. The proof is by induction on the Kemer index p associated to an H - T -ideal Γ (satisfying a Capelli identity). If $p = 0$ then $\Gamma = F^H \langle X \rangle$ and so $W = 0$. Suppose the theorem is true for any affine H -module algebra with H -Kemer index smaller than p . Denote by S_p the H - T -ideal generated by all H -Kemer polynomials corresponding to Γ , and let $\Gamma' = \Gamma + S_p$. It is clear that the H -Kemer index of Γ' is smaller than p . Hence, by the inductive hypothesis there is a representable H -module algebra A' having Γ' as its H - T -ideal of identities.

Let B_Γ be the representable H -module algebra constructed in the previous section. We'll show $\Gamma = id^H(A' \times B_\Gamma)$.

It is clear that $\Gamma \subset id^H(A' \times B_\Gamma)$ since Γ is contained in Γ' and by construction $\Gamma \subseteq id^H(B_\Gamma)$. Suppose there is $f \notin \Gamma$ with $f \in id^H(A' \times B_\Gamma) = id^H(A') \cap id^H(B_\Gamma)$. Since $f \in id^H(A') = \Gamma'$, we may assume $f \in S_p$. Using the Phoenix property 9.3, we obtain a Kemer polynomial f' (with at least $\mu + 1$ small sets) such that $f' \in (f)_H$. But by 10.4, $f \notin id^H(B_\Gamma)$ and this contradicts our previous assumption on f . This completes the proof. \square

11.1. Non affine case. Let H be any F -Hopf algebra. Denote by H_2 the Hopf algebra $H \otimes_F (FC_2)^*$, where C_2 is the additive group with two elements 0 and 1. An F -algebra W is an H_2 -module algebra if it is C_2 -graded H -module algebra such that the graded component W_0 and W_1 of W are stable under the action of H . We denote by G the Grassmann algebra over F , which is a C_2 -graded algebra. If W is an H_2 -module algebra

then we can define the C_2 -graded algebra $E(W) = W_0 \otimes E_0 \oplus W_1 \otimes E_1$. It also has an H structure given by

$$h \cdot (a_0 \otimes w_0 + a_1 \otimes w_1) = h(a_0) \otimes w_0 + h(a_1) \otimes w_1.$$

Therefore we obtain an H_2 -algebra.

It is possible to get an H_2 -module algebra from an H -module algebra: Let W be an H -module algebra. The algebra $W_E = W \otimes E$ is an H_2 -module algebra, where $(W_E)_0 = W \otimes E_0$ and $(W_E)_1 = W \otimes E_1$. The H -action is given by

$$h \cdot (w \otimes e) = (h \cdot w) \otimes e.$$

The H_2 -module algebra $F^{H_2} \{X\}$ can be considered as the H -module algebra $F^H \{Y, Z\}$, where Y and Z are countable sets of variables. the variables in Y are considered even and the ones in Z are odd. We identify $x_i \in X$ with $y_i + z_i$, thus for every $h \in H$ we have $x_i^h = y_i^h + z_i^h$. Denote by $L_{d,l}$ the affine H_2 -module algebra $F^H \{y_1, \dots, y_d, z_1, \dots, z_l\}$

The following is proven in [6].

Theorem 11.1. *If W is an H -module algebra which satisfy an ordinary PI f , then $id^H(W) = id^H(E(L))$ for $L = L_{d,l}/id^{H_2}(W_E)$, where d and l are determined by the degree of f .*

Since by theorem 1.1 L is H_2 -PI equivalent to a finite dimensional (over an extension field of F) H_2 -module algebra A , it is clear that:

Theorem 11.2 (theorem 1.2). *If W is an H -module algebra which satisfy an ordinary PI, then there is a finite dimensional H_2 -module algebra A over some extension field of F , such that $id^{H_2}(W_E) = id^{H_2}(A)$.*

12. SPECHT THEOREM FOR H -MODULE ALGEBRAS

In this section we prove theorem 1.3:

Theorem 12.1. *Suppose Γ is an H - T -ideal containing an ordinary identity, then there are $f_1, \dots, f_s \in \Gamma$ which H - T -generate Γ . Equivalently, if $\Gamma_1 \subseteq \Gamma_2 \subseteq \dots$ is an ascending chain of H - T -ideals containing an ordinary PI, then the chain stabilizes.*

Suppose $id^H(W_i) = \Gamma_i$. By theorem 11.1 $id^H(W_i) = id^H(E(L_{d,l}/id^{H_2}(W_i \otimes E)))$, where d and l are the same for all W_i . Moreover, it is clear that $id^{H_2}(W_1 \otimes E) \subseteq id^{H_2}(W_2 \otimes E) \subseteq \dots$, so

$$id^{H_2}(E(L_{d,l}/id^{H_2}(W_1 \otimes E))) \subseteq id^{H_2}(E(L_{d,l}/id^{H_2}(W_2 \otimes E))) \subseteq \dots$$

Therefore, it is enough to show:

Theorem 12.2. *If $\Gamma_1 \subseteq \Gamma_2 \subseteq \dots$ is an ascending chain of H - T -ideals of an affine H -module algebras containing an ordinary PI, then the chain stabilizes.*

By theorem 1.1 we can assume $id^H(A_i) = \Gamma_i$, where A_i is a product of H -basic algebras. Moreover, since the H -Kemer index is an order reversing function, the H -Kemer index is eventually stabilizes. Thus, we suppose from the beginning that all the A_i have the same H -Kemer index $p = (d, s)$. Write

$$A_i = \hat{A}_{i,1} \times \dots \times \hat{A}_{i,r_i} \times \tilde{A}_i,$$

where $\bar{A}_{i,1}, \dots, \bar{A}_{i,r_i}$ are H -basic of index p and \tilde{A}_i is a product of H -basic algebras of lower index. Since, due to [7], the number of non H -isomorphic H -semisimple algebras of dimension d is finite, by passing to a subsequence, we may also assume that there is a fixed set of H -semisimple algebras R_1, \dots, R_t such that:

$$\{R_1, \dots, R_t\} = \{\hat{A}_{i,1,ss}, \dots, \hat{A}_{i,r_i,ss}\}$$

for all i (here $\hat{A}_{i,r_i,ss}$ is the semisimple part of \hat{A}_{i,r_i}).

Let

$$C_{j,i} = \frac{\bar{C}_{j,i} = R_j * F^H \{X = \{x_1, \dots, x_s\}\}}{id_{C_{j,i}}^H(A_i) + \langle X \rangle_H^s}$$

for every $j = 1 \dots t$. Finally, write $C_i = C_{1,i} \times \dots \times C_{t,i}$.

Lemma 12.3. $id^H(C_i \times \tilde{A}_i) = id^H(A_i)$.

Proof. Clearly, $id^H(A_i) \subseteq id^H(C_i \times \tilde{A}_i)$. Since $id^H(A_i) = id^H(\hat{A}_{i,1} \times \dots \times \hat{A}_{i,r_i} \times \tilde{A}_i)$, it is enough to show that $id^H(C_i) \subseteq id^H(\hat{A}_{i,1} \times \dots \times \hat{A}_{i,r_i})$. So we prove that if $f = f(x_1, \dots, x_n)$ is a multilinear non-identity of some $\hat{A}_{i,j}$, (say with semisimple part equal to R_1) then f is also a non-identity of $C_{1,i}$. Indeed, choose a non-zero evaluation of f by elements of $\hat{A}_{i,j}$. Suppose $\bar{x}_1, \dots, \bar{x}_l$ are radical (so $l < s$) and the rest are semisimple (i.e. in R_1). Thus, $f(x_1, \dots, x_l, \bar{x}_{l+1}, \dots, \bar{x}_n)$ is not zero in $C_{1,i}$. \square

Lemma 12.4. For i large enough $C_i = C_{i+1} = \dots (= B)$.

Proof. It is clear that C_{i+1} is an H -epimorphic image of C_i , so $\dim_F C_i \geq \dim C_{i+1}$. However, all the C_i are finite dimensional, so for i large enough the sequence of dimensions stabilizes. Hence, C_{i+1} is H -isomorphic to C_i . \square

We are ready to conclude the proof of theorem 12.2 (and thus also of theorem 12.1). We are in the following situation:

$$id^H(B \times \tilde{A}_1) \subseteq id^H(B \times \tilde{A}_2) \subseteq \dots,$$

where the index of B is p and the index of \tilde{A}_i is smaller than p . Assume by induction on the H -Kemer index that theorem 12.2 holds for H - T -ideal of index smaller than p . Denote by I the H - T -ideal generated by all H -Kemer polynomials of B . Clearly the sequence

$$id^H(B \times \tilde{A}_1) + I \subseteq id^H(B \times \tilde{A}_2) + I \subseteq \dots$$

stabilizes (by induction). Moreover, for all i and j :

$$id^H(B \times \tilde{A}_i) \cap I = id^H(B \times \tilde{A}_j) \cap I$$

because $I \subseteq id^H(\tilde{A}_i) \cap id^H(\tilde{A}_j)$. Therefore, the original sequence also stabilizes.

13. THE EXPONENT OF H -MODULE ALGEBRAS

In this section we prove theorem 1.5.

Let W be an H -module algebra satisfying an ordinary PI. By theorem 1.2 we may assume that $W = G(A)$, where A is an H_2 -module finite dimensional algebra. Gordienko in [9] showed that theorem 1.5 holds when W is finite dimensional. So it will be enough to prove that $\exp^H(W) = \exp^{H_2}(A)$. The idea is to combine the following key theorem (of Gordienko) with the technique of Giambruno and Zaicev in [8] (Chapter 6.3).

Theorem 13.1 (Lemma 10 in [9]). *For almost all n there is an n -multilinear H_2 -polynomial $f(X_1, \dots, X_\mu, X_0) \notin \text{id}^{H_2}(A)$ such that $|X_0| < \alpha$ (for α not depended on n), $|X_1| = \dots = |X_\mu| = \exp^{H_2}(A)$ and f is alternating on each one of the X_i .*

The number d in the above theorem can be computed in the following fashion: Consider the H -Wedderburn-Malcev decomposition of A :

$$A = J(A) \oplus R_1 \times \dots \times R_q,$$

where the R_i are H_2 -simple. Then

$$\exp^{H_2}(A) = \max \left\{ \sum_{k=1}^t \dim R_{i_k} \mid R_{i_1} J(A) \cdots J(A) R_{i_t} \neq 0 \text{ and } i_1, \dots, i_t \text{ are distinct} \right\}.$$

Moreover, we know that A can be replaced by a product of H -basic algebras $A_1 \times \dots \times A_p$. It is obvious that $\exp^{H_2}(A) = \max_i \exp^{H_2}(A_i)$. Furthermore, the exponent of A_i (since A_i is full) is exactly $\alpha(A_i)$ - the first component of the H_2 -Kemer index of A_i . Thus, $\exp^{H_2}(A) = \alpha(A)$. Using the construction in lemma 6.6 and the previous theorem we obtain:

Theorem 13.2. *For almost all n there is an n -multilinear H_2 -Kemer polynomial $f = f(X_1, \dots, X_{\mu+s}, X_0) \notin \text{id}^{H_2}(A)$ such that $|X_0 \cup X_{\mu+1} \cup \dots \cup X_{\mu+s}| < \beta$ (for β not depended on n), X_1, \dots, X_μ are the small sets and $X_{\mu+1}, \dots, X_{\mu+s}$ are the big sets.*

Suppose $A = A_1 \times \dots \times A_r$ is a product of H_2 -simple algebras and suppose f from the previous theorem is an H_2 -Kemer polynomial of A . Therefore, f is also an H_2 -Kemer polynomial of one of the A_i , say A_1 . Denote by B the H_2 -algebra A_{ss} . We may assume that each $X_i \in \{X_0, \dots, X_{\mu+s}\}$ can be replaced by $Y_i \cup Z_i$, where Y_i is a set of even variables and Z_i of odd variables, such that the resulting polynomial is a non-identity of A_1 . Surely, $|Y_i| = \dim B_0 = d$ and $|Z_i| = \dim B_1 = l$ for $i = 1 \dots \mu$. Indeed, consider a non-zero substitution of f by elements from $B_0 \cup B_1 \cup J(A_1)$. Each big set of f has $\dim B + 1$ elements, so any non-zero substitution must include a radical element. There are s big sets and the nilpotency index of $J(A_1)$ is $s+1$, hence all the other substitutions of variables of f must be semisimple. Moreover, since f is alternating on each one of the small sets, we must substitute a full basis in each of them. Thus, d variables of each X_i must be even and the rest odd. Let us call the new polynomial also f .

Definition 13.3. Suppose $f = f(y_1, \dots, y_n, z_1, \dots, z_s)$ is a multilinear polynomial in $F^H \{Y, Z\}$. We may write

$$f = \sum a_{\sigma, W, \mathbf{h}} W_0 z_{\sigma(1)}^{h_1} W_1 z_{\sigma(2)}^{h_2} \cdots z_{\sigma(s)}^{h_s} W_s,$$

where the sum runs on $\sigma \in S_s$, $W = (W_0, \dots, W_s)$ - s -tuple of monomials (we allow the zero monomial: 1) in distinct variables from $\{y_1, \dots, y_n\}$ and $\mathbf{h} = (h_1, \dots, h_s) \in \{b_1, \dots, b_m\}^s$ (recall that b_1, \dots, b_m is a basis of H). Then \tilde{f} is defined to be the polynomial

$$\sum (-1)^\sigma a_{\sigma, W, \mathbf{h}} W_0 z_{\sigma(1)}^{h_1} W_1 z_{\sigma(2)}^{h_2} \cdots z_{\sigma(s)}^{h_s} W_s.$$

The following is folklore:

Lemma 13.4. *Suppose f is a multilinear polynomial in $F^H\{Y, Z\}$. Then the following holds:*

- (1) $f \in id^{H_2}(W) \iff \tilde{f} \in id^{H_2}(G(A))$.
- (2) $c_n^{H_2}(W) = c_n^{H_2}(G(W))$.

The next key Proposition relies on the representation theory of S_n . The reader is advised to review section 7.1 for a notation reminder.

Proposition 13.5. *Let f be the polynomial from the previous theorem and let $g = \tilde{f}$. Then for $\lambda' = (\mu^d)$ and $\lambda'' = (l^\mu)$, there are $T_{\lambda'}$ and $T_{\lambda''}$ such that $e_{T_{\lambda'}} e_{T_{\lambda''}} g \notin id^{H_2}(E(A))$.*

Proof. Since $g \notin id^{H_2}(E(A))$, there is some $\lambda' = (\lambda'_1, \dots) \vdash d\mu$ and $T_{\lambda'}$ indexed by variables from $Y = Y_1 \cup \dots \cup Y_\mu$ such that $e_{T_{\lambda'}} g \notin id^{H_2}(E(A))$. We claim that λ' is of the shape (μ^d) :

Write

$$e_1 = \sum_{\sigma \in \mathcal{R}_{T_{\lambda'}}} \sigma, \quad e_2 = \sum_{\sigma \in \mathcal{C}_{T_{\lambda'}}} (-1)^\sigma \sigma,$$

so $e_{T_{\lambda'}} = e_1 e_2$.

If $\lambda'_1 > d$, then $e_{T_{\lambda'}} g$ is symmetric on (at least) $\mu + 1$ variables from Y . Thus, for every $\sigma \in \mathcal{C}_{T_{\lambda'}}$ at least two of them must fall in the same $\sigma(Y_i)$. However, σg is alternating on $\sigma(Y_i)$, so $e_1 \sigma g = 0 \Rightarrow e_{T_{\lambda'}} g = e_1 e_2 g = 0$ - contradiction.

Suppose $h(\lambda') > d$, where $h\{\lambda'\}$ is the height of $T_{\lambda'}$. Therefore, $e_{T_{\lambda'}} g$ is alternating on a $d + 1$ subset Y' of Y . Thus $\widetilde{e_{T_{\lambda'}} g} = e_{T_{\lambda'}} \tilde{g} = e_{T_{\lambda'}} f$ is also alternating on Y' . On any non-zero substitution of f (on every A_i) the radical values appear only in $X_{\mu+1}, \dots, X_{\mu+t}$. Hence, the non-zero evaluations of Y' must consist of semisimple (even) elements. However, the dimension of the even semisimple part is d , thus $e_2 e_1 f \in id^{H_2}(A)$, hence $e_{T_{\lambda'}} g \in id^{H_2}(E(A))$ - contradiction. All in all, λ' must be equal to (μ^d) .

There is some $\lambda'' = (\lambda''_1, \dots) \vdash l\mu$ and $T_{\lambda''}$ indexed by variables from $Z = Z_1 \cup \dots \cup Z_\mu$ such that $e_{T_{\lambda''}} g \notin id^{H_2}(G(A))$. As before, we plan to demonstrate that λ'' has the shape (μ^d) .

Suppose $h(\lambda'') > \mu$. Write $e_1 = \sum_{\sigma \in \mathcal{R}_{T_{\lambda''}}} \sigma$ and $e_2 = \sum_{\sigma \in \mathcal{C}_{T_{\lambda''}}} (-1)^\sigma \sigma$ so that $e_{T_{\lambda''}} = e_1 e_2$. $e_2 g$ is alternating on (at least) $\mu + 1$ variables from Z . Thus, at least two of them must fall in some Z_i . Since $e_2 g$ is also symmetric on each Z_i , $e_2 g = 0$. So $e_{T_{\lambda''}} g = 0$.

Suppose $\lambda''_1 > l$. Then, $e_{T_{\lambda''}} g$ is symmetric on $l + 1$ elements of Z , say Z' . Hence $\widetilde{e_{T_{\lambda''}} g} = \tilde{e}_{T_{\lambda''}} \tilde{g}$ is alternating on Y' . where

$$\tilde{e}_{T_{\lambda''}} = \sum_{\tau \in \mathcal{R}_{T_{\lambda''}}} \sum_{\sigma \in \mathcal{C}_{T_{\lambda''}}} (-1)^\sigma \sigma \tau = \sum_{\tau \in \mathcal{R}_{T_{\lambda''}}} \sum_{\sigma \in \mathcal{C}_{T_{\lambda''}}} (-1)^\sigma \tau \sigma.$$

As before, this implies that $\widetilde{e_{T_{\lambda''}}g} \in id^{H_2}(A)$, hence $e_{T_{\lambda''}}g \in id^{H_2}(E(A))$ - contradiction. All in all, λ'' must be equal to (l^μ) .

Because, $e_{T_{\lambda'}}$ and $e_{T_{\lambda''}}$ act on distinct sets of variables, we conclude

$$e_{T_{\lambda'}}e_{T_{\lambda''}}g \notin id^H(E(A)).$$

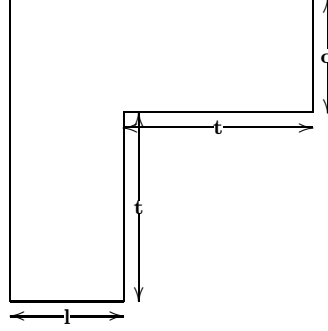
□

Corollary 13.6. *If we replace the graded variables of g by non graded ones, we obtain an H -polynomial (which we continue to denote by g) which is not in $id^H(E(A))$. Moreover, $e_{T_{\lambda'}}e_{T_{\lambda''}}g \notin id^H(E(A))$.*

Definition 13.7. Let $t > 0$ be an integer number. $h(d, l, t)$ is the partition of $d(l+t)+tl$ given by:

$$\underbrace{(l+t, \dots, l+t)}_{d \text{ times}}, \underbrace{(l, \dots, l)}_{t \text{ times}}.$$

The corresponding tableau looks like:



Corollary 13.8. *For any $\mu > 0$, there exist a partition λ of $\mu(d+l)$ such that*

$$h(d, l, \mu - d - l) \leq \lambda \leq h(d, l, \mu)$$

and for $n = \mu(d+l) + \beta$ the space $P_n^H / P_n^H \cap id^H(G(A))$ contains an irreducible $S_{|\lambda|}$ -module corresponding to λ . Furthermore,

$$c_n^H(G(A)) \geq Cn^\gamma (\exp^{H_2}(A))^n,$$

for some numbers C and γ .

Proof. The first part follows from the previous corollary by means of the Littlewood-Richardson rule (see Theorem 2.3.9 in [8]). The second part follows from the first part and Lemma 6.2.5 in [8]. □

To finish the proof of theorem 1.5, it is suffice to establish that $c_n^H(E(A)) \leq c_n^{H_2}(A)$. Indeed, for every H_2 -module algebra A ,

$$\frac{P_n^H}{P_n^H \cap id^H(A)} \hookrightarrow \frac{P_n^{H_2}}{P_n^{H_2} \cap id^{H_2}(A)},$$

where the map is induced by $x_i \rightarrow y_i + z_i$.

Thus, $c_n^H(E(A)) \leq c_n^{H_2}(E(A))$. Since $c_n^{H_2}(E(A)) = c_n^{H_2}(A)$, we obtain $c_n^H(E(A)) \leq c_n^{H_2}(A)$.

REFERENCES

- [1] Uzi Vishne Alexei Belov-Kanel, Louis Rowen. Specht’s problem for associative affine algebras over commutative noetherian rings. To appear in the Transactions of the AMS.
- [2] Yaakov Karasik Alexei Kanel-Belov and Louis Halle Rowen. Computational aspects of polynomial identities, volume i. In print.
- [3] Eli Aljadeff and Alexei Kanel-Belov. Representability and Specht problem for G -graded algebras. *Adv. Math.*, 225(5):2391–2428, 2010.
- [4] Eli Aljadeff and Yaakov Karasik. Kemer’s theorem for affine pi algebras over a field of characteristic zero. Preprint.
- [5] K. I. Beidar. On a. i. mal’tsev’s theorems on matrix representations of algebras. *Uspekhi Mat. Nauk*, 41(5(251)):161–162, 1986.
- [6] Allan Berele and Jeffrey Bergen. P.I. algebras with Hopf algebra actions. *J. Algebra*, 214(2):636–651, 1999.
- [7] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. On fusion categories. *Ann. of Math. (2)*, 162(2):581–642, 2005.
- [8] Antonio Giambruno and Mikhail Zaicev. *Polynomial identities and asymptotic methods*, volume 122 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.
- [9] A. S. Gordienko. Amitsur’s conjecture for associative algebras with a generalized Hopf action. *J. Pure Appl. Algebra*, 217(8):1395–1411, 2013.
- [10] Alexei Kanel-Belov and Louis Halle Rowen. *Computational aspects of polynomial identities*, volume 9 of *Research Notes in Mathematics*. A K Peters, Ltd., Wellesley, MA, 2005.
- [11] A. R. Kemer. Varieties and Z_2 -graded algebras. *Izv. Akad. Nauk SSSR Ser. Mat.*, 48(5):1042–1059, 1984.
- [12] V. Linchenko, S. Montgomery, and L. W. Small. Stable Jacobson radicals and semiprime smash products. *Bull. London Math. Soc.*, 37(6):860–872, 2005.
- [13] Wilhelm Specht. Gesetze in Ringen. I. *Math. Z.*, 52:557–589, 1950.
- [14] D. Ştefan and F. Van Oystaeyen. The Wedderburn-Malcev theorem for comodule algebras. *Comm. Algebra*, 27(8):3569–3581, 1999.
- [15] Irina Sviridova. Identities of pi-algebras graded by a finite abelian group. *Comm. Algebra*, 39(9):3462–3490, 2011.
- [16] Irina Sviridova. Finitely generated algebras with involution and their identities. *J. Algebra*, 383:144–167, 2013.

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